Transcendence of special values of Goss L-functions attached to Drinfeld modules

Changningphaabi Namoijam, NTHU

2021 TMS Annual Meeting, January 17-18, 2022

(joint work with Oğuz Gezmiș)

Overview

1. Classical case
2. Goss L-functions and results

Section 1

Classical case

L-functions

- The Riemann zeta function $\zeta(\mathfrak{s})$ is given by

$$\zeta(\mathfrak{s}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathfrak{s}}}, \quad \text{where } \mathfrak{s} \in \mathbb{C} \text{ satisfying } \Re(\mathfrak{s}) > 1.$$

L-functions

- The Riemann zeta function $\zeta(\mathfrak{s})$ is given by

$$\zeta(\mathfrak{s}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathfrak{s}}}, \quad \text{where } \mathfrak{s} \in \mathbb{C} \text{ satisfying } \Re(\mathfrak{s}) > 1.$$

- When s is a positive integer, $\zeta(2s)$ is transcendental over $\mathbb{Q}$.

L-functions

- The Riemann zeta function $\zeta(\mathfrak{s})$ is given by

$$\zeta(\mathfrak{s}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathfrak{s}}}, \quad \text{where } \mathfrak{s} \in \mathbb{C} \text{ satisfying } \Re(\mathfrak{s}) > 1.$$

- When s is a positive integer, $\zeta(2s)$ is transcendental over $\mathbb{Q}$.
- When $s \in \mathbb{Z}_{\geq 2}$, it is expected that $\zeta(s)$ is transcendental over $\mathbb{Q}$.

Let E be an elliptic curve defined over $\mathbb{Z}$. For a prime $p \in \mathbb{Z}$, let E_p denote the reduction of E modulo p.

Let E be an elliptic curve defined over $\mathbb{Z}$. For a prime $p \in \mathbb{Z}$, let E_p denote the reduction of E modulo p. Set $a_p := p + 1 - \#E(\mathbb{F}_p)$, and define the polynomial $L_p(X) \in \mathbb{Z}[X]$ given by

Let E be an elliptic curve defined over $\mathbb{Z}$. For a prime $p \in \mathbb{Z}$, let E_p denote the reduction of E modulo p. Set $a_p := p + 1 - \#E(\mathbb{F}_p)$, and define the polynomial $L_p(X) \in \mathbb{Z}[X]$ given by

$$L_p(X) := \begin{cases} 1 - a_p X + pX^2 & \text{if } E \text{ has good reduction at } p \\ 1 - X & \text{if } E \text{ has split mult. reduction at } p \\ 1 + X & \text{if } E \text{ has nonsplit mult. reduction at } p \\ 1 & \text{if } E \text{ has additive reduction at } p \end{cases}.$$

Let E be an elliptic curve defined over $\mathbb{Z}$. For a prime $p \in \mathbb{Z}$, let E_p denote the reduction of E modulo p. Set $a_p := p + 1 - \#E(\mathbb{F}_p)$, and define the polynomial $L_p(X) \in \mathbb{Z}[X]$ given by

$$L_p(X) := \begin{cases} 1 - a_p X + pX^2 & \text{if } E \text{ has good reduction at } p \\ 1 - X & \text{if } E \text{ has split mult. reduction at } p \\ 1 + X & \text{if } E \text{ has nonsplit mult. reduction at } p \\ 1 & \text{if } E \text{ has additive reduction at } p \end{cases}.$$

For any $\mathfrak{s} \in \mathbb{C}$ with $\Re(\mathfrak{s}) > 3/2$, we define the L-function $L(E, \mathfrak{s})$ of E by

$$L(E, \mathfrak{s}) = \prod_p L_p(p^{-\mathfrak{s}})^{-1}.$$

where the product runs over all prime numbers.

Let E be an elliptic curve defined over $\mathbb{Z}$. For a prime $p \in \mathbb{Z}$, let E_p denote the reduction of E modulo p. Set $a_p := p + 1 - \#E(\mathbb{F}_p)$, and define the polynomial $L_p(X) \in \mathbb{Z}[X]$ given by

$$L_p(X) := \begin{cases} 1 - a_p X + pX^2 & \text{if } E \text{ has good reduction at } p \\ 1 - X & \text{if } E \text{ has split mult. reduction at } p \\ 1 + X & \text{if } E \text{ has nonsplit mult. reduction at } p \\ 1 & \text{if } E \text{ has additive reduction at } p \end{cases}.$$

For any $\mathfrak{s} \in \mathbb{C}$ with $\Re(\mathfrak{s}) > 3/2$, we define the L-function $L(E, \mathfrak{s})$ of E by

$$L(E, \mathfrak{s}) = \prod_p L_p(p^{-\mathfrak{s}})^{-1}.$$

where the product runs over all prime numbers.

- Bloch and Beilinson calculated the value of $L(E, 2)$ for particular elliptic curves, and Beilinson and Deligne gave explicit conjectures about the special values of $L(E, \mathfrak{s})$.

Let E be an elliptic curve defined over $\mathbb{Z}$. For a prime $p \in \mathbb{Z}$, let E_p denote the reduction of E modulo p. Set $a_p := p + 1 - \#E(\mathbb{F}_p)$, and define the polynomial $L_p(X) \in \mathbb{Z}[X]$ given by

$$L_p(X) := \begin{cases} 1 - a_p X + pX^2 & \text{if } E \text{ has good reduction at } p \\ 1 - X & \text{if } E \text{ has split mult. reduction at } p \\ 1 + X & \text{if } E \text{ has nonsplit mult. reduction at } p \\ 1 & \text{if } E \text{ has additive reduction at } p \end{cases}.$$

For any $\mathfrak{s} \in \mathbb{C}$ with $\Re(\mathfrak{s}) > 3/2$, we define the L-function $L(E, \mathfrak{s})$ of E by

$$L(E, \mathfrak{s}) = \prod_p L_p(p^{-\mathfrak{s}})^{-1}.$$

where the product runs over all prime numbers.

- Bloch and Beilinson calculated the value of $L(E, 2)$ for particular elliptic curves, and Beilinson and Deligne gave explicit conjectures about the special values of $L(E, \mathfrak{s})$.
- Transcendence of special values of $L(E, \mathfrak{s})$ over $\mathbb{Q}$ is not known.

Section 2

Goss L-functions and results

Notation

$\mathbb{F}_q :=$ finite field, with q a power of a prime p

$\mathbf{A} := \mathbb{F}_q[t]$, the polynomial ring in the variable t over $\mathbb{F}_q$

$A := \mathbb{F}_q[\theta]$, the polynomial ring in the variable θ over $\mathbb{F}_q$

$A_+ :=$ the set of monic polynomials of A

$K := \mathbb{F}_q(\theta)$, rational functions in the variable θ over $\mathbb{F}_q$

$K_\infty := \mathbb{F}_q((1/\theta))$, the completion of K with respect to the fixed absolute value $|\cdot|_\infty$ at the infinite place normalized so that $|\theta|_\infty = q$

$\mathbb{C}_\infty :=$ completion of an algebraic closure of K_∞

$\overline{K} :=$ algebraic closure of K inside $\mathbb{C}_\infty$

- For any $a \in \mathbb{C}_\infty$ and $i \in \mathbb{Z}$, we define

$$\tau^i(a) := a^{(i)} := a^{q^i}.$$

- For any $a \in \mathbb{C}_\infty$ and $i \in \mathbb{Z}$, we define

$$\tau^i(a) := a^{(i)} := a^{q^i}.$$

Let $m, \ell \in \mathbb{Z}_{\geq 1}$. For a matrix $B = (b_{j,k}) \in \mathrm{Mat}_{m\times\ell}(\mathbb{C}_\infty)$, we set $B^{(i)} = (b_{j,k}^{(i)})$.

- For any $a \in \mathbb{C}_\infty$ and $i \in \mathbb{Z}$, we define

$$\tau^i(a) := a^{(i)} := a^{q^i}.$$

Let $m, \ell \in \mathbb{Z}_{\geq 1}$. For a matrix $B = (b_{j,k}) \in \mathrm{Mat}_{m \times \ell}(\mathbb{C}_\infty)$, we set $B^{(i)} = (b_{j,k}^{(i)})$.

- Define the non-commutative polynomial ring $L[\tau]$ subject to the condition

$$\tau c = c^{(1)} \tau, \quad c \in L.$$

- For any $a \in \mathbb{C}_\infty$ and $i \in \mathbb{Z}$, we define

$$\tau^i(a) := a^{(i)} := a^{q^i}.$$

  Let $m, \ell \in \mathbb{Z}_{\geq 1}$. For a matrix $B = (b_{j,k}) \in \mathrm{Mat}_{m\times\ell}(\mathbb{C}_\infty)$, we set $B^{(i)} = (b_{j,k}^{(i)})$.
- Define the non-commutative polynomial ring $L[\tau]$ subject to the condition

$$\tau c = c^{(1)}\tau, \quad c \in L.$$

- We let $\mathrm{Mat}_{m\times\ell}(L)[\tau]$ be the set of polynomials of τ with coefficients in $\mathrm{Mat}_{m\times\ell}(L)$.

- For any $a \in \mathbb{C}_\infty$ and $i \in \mathbb{Z}$, we define

$$\tau^i(a) := a^{(i)} := a^{q^i}.$$

Let $m, \ell \in \mathbb{Z}_{\geq 1}$. For a matrix $B = (b_{j,k}) \in \mathrm{Mat}_{m \times \ell}(\mathbb{C}_\infty)$, we set $B^{(i)} = (b_{j,k}^{(i)})$.

- Define the non-commutative polynomial ring $L[\tau]$ subject to the condition

$$\tau c = c^{(1)}\tau, \quad c \in L.$$

- We let $\mathrm{Mat}_{m \times \ell}(L)[\tau]$ be the set of polynomials of τ with coefficients in $\mathrm{Mat}_{m \times \ell}(L)$.
- When $m = \ell$, define the ring $\mathrm{Mat}_m(L)[[\tau]]$ of power series of τ with coefficients in $\mathrm{Mat}_m(L) := \mathrm{Mat}_{m \times m}(L)$ subject to the condition

$$\tau B = B^{(1)}\tau, \quad B \in \mathrm{Mat}_m(L).$$

- For any $a \in \mathbb{C}_\infty$ and $i \in \mathbb{Z}$, we define

$$\tau^i(a) := a^{(i)} := a^{q^i}.$$

Let $m, \ell \in \mathbb{Z}_{\geq 1}$. For a matrix $B = (b_{j,k}) \in \mathrm{Mat}_{m \times \ell}(\mathbb{C}_\infty)$, we set $B^{(i)} = (b_{j,k}^{(i)})$.

- Define the non-commutative polynomial ring $L[\tau]$ subject to the condition

$$\tau c = c^{(1)} \tau, \quad c \in L.$$

- We let $\mathrm{Mat}_{m \times \ell}(L)[\tau]$ be the set of polynomials of τ with coefficients in $\mathrm{Mat}_{m \times \ell}(L)$.
- When $m = \ell$, define the ring $\mathrm{Mat}_m(L)[[\tau]]$ of power series of τ with coefficients in $\mathrm{Mat}_m(L) := \mathrm{Mat}_{m \times m}(L)$ subject to the condition

$$\tau B = B^{(1)} \tau, \quad B \in \mathrm{Mat}_m(L).$$

Also let $\mathrm{Mat}_m(L)[\tau] \subset \mathrm{Mat}_m(L)[[\tau]]$ to be the subring of polynomials of τ.

Let R be an A-subalgebra of $\mathbb{C}_\infty$. A Drinfeld $\mathbf{A}$-module ϕ defined over R of rank r is an $\mathbb{F}_q$-algebra homomorphism $\phi : \mathbf{A} \to R[\tau]$ defined by

$$\phi(t) := \theta + c_1\tau + \cdots + c_r\tau^r, \quad c_r \neq 0.$$

Let R be an A-subalgebra of $\mathbb{C}_\infty$. A Drinfeld $\mathbf{A}$-module ϕ defined over R of rank r is an $\mathbb{F}_q$-algebra homomorphism $\phi : \mathbf{A} \to R[\tau]$ defined by

$$\phi(t) := \theta + c_1\tau + \cdots + c_r\tau^r, \quad c_r \neq 0.$$

- We say that a Drinfeld $\mathbf{A}$-module φ defined over A is isomorphic to ϕ over K if there exists $c \in K^\times$ we have

$$c\varphi(t) = \phi(t)c.$$

Let R be an A-subalgebra of $\mathbb{C}_\infty$. A Drinfeld $\mathbf{A}$-module ϕ defined over R of rank r is an $\mathbb{F}_q$-algebra homomorphism $\phi : \mathbf{A} \to R[\tau]$ defined by

$$\phi(t) := \theta + c_1\tau + \cdots + c_r\tau^r, \quad c_r \neq 0.$$

- We say that a Drinfeld $\mathbf{A}$-module φ defined over A is isomorphic to ϕ over K if there exists $c \in K^\times$ we have

$$c\varphi(t) = \phi(t)c.$$

- Let φ be given by

$$\varphi(t) := \theta + b_1\tau + \cdots + b_r\tau^r,$$

where $b_1, \ldots, b_r \in A$, $b_r \neq 0$.

Let R be an A-subalgebra of $\mathbb{C}_\infty$. A Drinfeld $\mathbf{A}$-module ϕ defined over R of rank r is an $\mathbb{F}_q$-algebra homomorphism $\phi : \mathbf{A} \to R[\tau]$ defined by

$$\phi(t) := \theta + c_1\tau + \cdots + c_r\tau^r, \quad c_r \neq 0.$$

- We say that a Drinfeld $\mathbf{A}$-module φ defined over A is isomorphic to ϕ over K if there exists $c \in K^\times$ we have

$$c\varphi(t) = \phi(t)c.$$

- Let φ be given by

$$\varphi(t) := \theta + b_1\tau + \cdots + b_r\tau^r,$$

  where $b_1, \ldots, b_r \in A$, $b_r \neq 0$.
- For an irreducible polynomial $\beta \in A_+$, let $\mathbb{F}_\beta$ denote the field $A/(\beta)$, and let $\overline{\varphi}$ be the Drinfeld $\mathbf{A}$-module given by reduction modulo β,

$$\overline{\varphi}(t) = \overline{\theta} + \overline{b}_1\tau + \cdots + \overline{b}_{r_0}\tau^{r_0} \quad , \overline{b}_{r_0} \neq 0.$$

Let R be an A-subalgebra of $\mathbb{C}_\infty$. A Drinfeld $\mathbf{A}$-module ϕ defined over R of rank r is an $\mathbb{F}_q$-algebra homomorphism $\phi : \mathbf{A} \to R[\tau]$ defined by

$$\phi(t) := \theta + c_1\tau + \cdots + c_r\tau^r, \quad c_r \neq 0.$$

- We say that a Drinfeld $\mathbf{A}$-module φ defined over A is isomorphic to ϕ over K if there exists $c \in K^\times$ we have

$$c\varphi(t) = \phi(t)c.$$

- Let φ be given by

$$\varphi(t) := \theta + b_1\tau + \cdots + b_r\tau^r,$$

  where $b_1, \ldots, b_r \in A$, $b_r \neq 0$.

- For an irreducible polynomial $\beta \in A_+$, let $\mathbb{F}_\beta$ denote the field $A/(\beta)$, and let $\overline{\varphi}$ be the Drinfeld $\mathbf{A}$-module given by reduction modulo β,

$$\overline{\varphi}(t) = \overline{\theta} + \overline{b}_1\tau + \cdots + \overline{b}_{r_0}\tau^{r_0} \quad , \overline{b}_{r_0} \neq 0.$$

We say that ϕ has good reduction at β if there exists a Drinfeld $\mathbf{A}$-module φ defined over A and isomorphic to ϕ such that $r_0 = r$.

- Let $\mathbb{F}_\beta^{\mathrm{sep}}$ be a fixed separable closure of $\mathbb{F}_\beta$. Let $\boldsymbol{v} \in \mathbf{A}$ be an irreducible polynomial, and let $\boldsymbol{v}|_{t=\theta} = v \in A$. Suppose that $v \neq \beta$. Define

$$\overline{\varphi}[v^n] := \{f \in \overline{\varphi}(\mathbb{F}_\beta^{\mathrm{sep}}) \mid \boldsymbol{v}^n \cdot f = 0\}.$$

- Let $\mathbb{F}_\beta^{\mathrm{sep}}$ be a fixed separable closure of $\mathbb{F}_\beta$. Let $\boldsymbol{v} \in \mathbf{A}$ be an irreducible polynomial, and let $\boldsymbol{v}|_{t=\theta} = v \in A$. Suppose that $v \neq \beta$. Define

$$\overline{\varphi}[v^n] := \{f \in \overline{\varphi}(\mathbb{F}_\beta^{\mathrm{sep}}) \mid \boldsymbol{v}^n \cdot f = 0\}.$$

Define the v*-adic Tate module of* $\overline{\varphi}$ by

$$T_v(\overline{\varphi}) := \varprojlim \overline{\varphi}[v^n].$$

- Let S be the set of primes in A_+ where ϕ does not have good reduction.

- Let S be the set of primes in A_+ where ϕ does not have good reduction.
- For $\beta \notin S$ prime, let $P_\beta(x)$ be the characteristic polynomial of the $q^{\deg_\theta(\beta)}$-th power Frobenius map $\tau^{\deg_\theta(\beta)}$ on the Tate module $T_v(\overline{\varphi})$. $P_\beta(x)$ does not depend on v and $P_\beta(x) \in A[x]$.

- Let S be the set of primes in A_+ where ϕ does not have good reduction.
- For $\beta \notin S$ prime, let $P_\beta(x)$ be the characteristic polynomial of the $q^{\deg_\theta(\beta)}$-th power Frobenius map $\tau^{\deg_\theta(\beta)}$ on the Tate module $T_v(\overline{\varphi})$. $P_\beta(x)$ does not depend on v and $P_\beta(x) \in A[x]$.
- For $\beta \notin S$ prime, let

$$Q_\beta(x) = x^r P_\beta(1/x)$$

be the reciprocal polynomial of $P_\beta(x)$.

- The *Goss L-function corresponding to ϕ* is defined as

$$L(\phi, s) := \alpha_{\phi,s} \prod_{\substack{\beta \notin S, \beta \in A_+ \\ \beta \text{ prime}}} Q_\beta(\beta^{-s})^{-1}, \quad s \in \mathbb{Z}.$$

- Here $\alpha_{\phi,s}$ is the product of local factors $\prod_{f \in S} Q'_f(f^{-s})^{-1}$, and is an element of $K^\times$ due to Gardeyn.

- The *Goss L-function corresponding to ϕ* is defined as

$$L(\phi, s) := \alpha_{\phi,s} \prod_{\substack{\beta \notin S, \beta \in A_+ \\ \beta \text{ prime}}} Q_\beta(\beta^{-s})^{-1}, \quad s \in \mathbb{Z}.$$

- Here $\alpha_{\phi,s}$ is the product of local factors $\prod_{f \in S} Q'_f(f^{-s})^{-1}$, and is an element of $K^\times$ due to Gardeyn.
- For example, if ϕ is the Carlitz module C, that is $C_t = \theta + \tau$, then for $n \geq 2$ the *Carlitz zeta values at $n-1$* is

$$L(C, n) = \sum_{a \in A_+} \frac{1}{a^{n-1}} \in K_\infty,$$

- The *Goss L-function corresponding to ϕ* is defined as

$$L(\phi, s) := \alpha_{\phi,s} \prod_{\substack{\beta \notin S, \beta \in A_+ \\ \beta \text{ prime}}} Q_\beta(\beta^{-s})^{-1}, \quad s \in \mathbb{Z}.$$

- Here $\alpha_{\phi,s}$ is the product of local factors $\prod_{f \in S} Q'_f(f^{-s})^{-1}$, and is an element of $K^\times$ due to Gardeyn.
- For example, if ϕ is the Carlitz module C, that is $C_t = \theta + \tau$, then for $n \geq 2$ the *Carlitz zeta values at $n-1$* is

$$L(C, n) = \sum_{a \in A_+} \frac{1}{a^{n-1}} \in K_\infty,$$

and Yu proved that $L(C, n)$ is transcendental over $\overline{K}$.

Theorem (Gezmiş-N., 2021)

Let n be a positive integer and ϕ be a Drinfeld $\mathbf{A}$-module defined over K. Then, $L(\phi, n)$ is transcendental over $\overline{K}$.

Definition

Let L be a subfield of $\mathbb{C}_\infty$. A *t-module of dimension $s \in \mathbb{Z}_{\geq 1}$ defined over L* is a tuple $G = (\mathbb{G}_{a/L}^s, \psi)$ consisting of the s-dimensional additive algebraic group $\mathbb{G}_{a/L}^s$ over L and an $\mathbb{F}_q$-algebra homomorphism $\psi : \mathbf{A} \to \mathrm{Mat}_s(L)[\tau]$ given by

$$\psi(t) = A_0 + A_1\tau + \cdots + A_m\tau^m$$

for some $m \in \mathbb{Z}_{\geq 0}$ so that $\mathrm{d}_\psi(t) := A_0 = \theta\,\mathrm{I}_s + N$ where I_s is the $s \times s$ identity matrix and N is a nilpotent matrix.

Definition

Let L be a subfield of $\mathbb{C}_\infty$. A *t-module of dimension $s \in \mathbb{Z}_{\geq 1}$ defined over L* is a tuple $G = (\mathbb{G}_{a/L}^s, \psi)$ consisting of the s-dimensional additive algebraic group $\mathbb{G}_{a/L}^s$ over L and an $\mathbb{F}_q$-algebra homomorphism $\psi : \mathbf{A} \to \mathrm{Mat}_s(L)[\tau]$ given by

$$\psi(t) = A_0 + A_1\tau + \cdots + A_m\tau^m$$

for some $m \in \mathbb{Z}_{\geq 0}$ so that $\mathrm{d}_\psi(t) := A_0 = \theta\,\mathrm{I}_s + N$ where I_s is the $s \times s$ identity matrix and N is a nilpotent matrix.

Set $\mathrm{Lie}(G)(L) := \mathrm{Mat}_{s\times 1}(L)$ and equip it with the $\mathbf{A}$-module structure given by

$$t \cdot x := \mathrm{d}_\psi(t)x = A_0 x, \quad x \in \mathrm{Lie}(G)(L).$$

Definition

Let L be a subfield of $\mathbb{C}_\infty$. A *t-module of dimension $s \in \mathbb{Z}_{\geq 1}$ defined over L* is a tuple $G = (\mathbb{G}_{a/L}^s, \psi)$ consisting of the s-dimensional additive algebraic group $\mathbb{G}_{a/L}^s$ over L and an $\mathbb{F}_q$-algebra homomorphism $\psi : \mathbf{A} \to \mathrm{Mat}_s(L)[\tau]$ given by

$$\psi(t) = A_0 + A_1\tau + \cdots + A_m\tau^m$$

for some $m \in \mathbb{Z}_{\geq 0}$ so that $\mathrm{d}_\psi(t) := A_0 = \theta\,\mathrm{I}_s + N$ where I_s is the $s \times s$ identity matrix and N is a nilpotent matrix.

Set $\mathrm{Lie}(G)(L) := \mathrm{Mat}_{s\times 1}(L)$ and equip it with the $\mathbf{A}$-module structure given by

$$t \cdot x := \mathrm{d}_\psi(t)x = A_0 x, \quad x \in \mathrm{Lie}(G)(L).$$

Also define $G(L) := \mathrm{Mat}_{s\times 1}(L)$ whose $\mathbf{A}$-module structure is given by

$$t \cdot x := \psi(t)x = A_0 x + A_1 x^{(1)} + \cdots + A_m x^{(m)}, \quad x \in G(L).$$

- There exists a unique infinite series $\mathrm{Exp}_G := \sum_{i\geq 0} \alpha_i \tau^i \in \mathrm{Mat}_s(L)[[\tau]]$ satisfying $\alpha_0 = \mathrm{I}_s$ and

$$\mathrm{Exp}_G \, \mathrm{d}_\psi(t) = \psi(t) \, \mathrm{Exp}_G \, .$$

- There exists a unique infinite series $\operatorname{Exp}_G := \sum_{i\geq 0} \alpha_i \tau^i \in \operatorname{Mat}_s(L)[[\tau]]$ satisfying $\alpha_0 = \mathrm{I}_s$ and

$$\operatorname{Exp}_G \mathrm{d}_\psi(t) = \psi(t) \operatorname{Exp}_G .$$

- It defines an everywhere convergent function $\operatorname{Exp}_G : \operatorname{Lie}(G)(\mathbb{C}_\infty) \to G(\mathbb{C}_\infty)$, *the exponential function of G*, given by $\operatorname{Exp}_G(x) = \sum_{i\geq 0} \alpha_i x^{(i)}$ for any $x \in \operatorname{Lie}(G)(\mathbb{C}_\infty)$.

- There exists a unique infinite series $\mathrm{Exp}_G := \sum_{i\geq 0} \alpha_i \tau^i \in \mathrm{Mat}_s(L)[[\tau]]$ satisfying $\alpha_0 = \mathrm{I}_s$ and

$$\mathrm{Exp}_G \, \mathrm{d}_\psi(t) = \psi(t) \, \mathrm{Exp}_G \, .$$

- It defines an everywhere convergent function $\mathrm{Exp}_G : \mathrm{Lie}(G)(\mathbb{C}_\infty) \to G(\mathbb{C}_\infty)$, *the exponential function of* G, given by $\mathrm{Exp}_G(x) = \sum_{i\geq 0} \alpha_i x^{(i)}$ for any $x \in \mathrm{Lie}(G)(\mathbb{C}_\infty)$.
- We set the $\mathbf{A}$-module $\Lambda_G := \mathrm{Ker}(\mathrm{Exp}_G) \subset \mathrm{Lie}(G)(\mathbb{C}_\infty)$, and a non-zero element of Λ_G is called a *a period of* G.

- Recall ϕ the Drinfeld $\mathbf{A}$-module defined over K defined by

$$\phi(t) = \theta + c_1\tau + \cdots + c_r\tau^r, \quad \text{where } c_r \neq 0.$$

- Recall ϕ the Drinfeld $\mathbf{A}$-module defined over K defined by
$$\phi(t) = \theta + c_1\tau + \cdots + c_r\tau^r, \quad \text{where } c_r \neq 0.$$

- For $n \geq 1$, let $G_n := (\mathbb{G}_{a/K}^{rn+r-1}, \phi_n)$ be the t-module given by
$$\phi_n(t) := \theta \operatorname{Id}_{rn+r-1} + N + c_r^{-1} E\tau,$$

- Recall ϕ the Drinfeld $\mathbf{A}$-module defined over K defined by
$$\phi(t) = \theta + c_1\tau + \cdots + c_r\tau^r, \quad \text{where } c_r \neq 0.$$

- For $n \geq 1$, let $G_n := (\mathbb{G}_{a/K}^{rn+r-1}, \phi_n)$ be the t-module given by
$$\phi_n(t) := \theta \,\mathrm{Id}_{rn+r-1} + N + c_r^{-1} E\tau,$$

where

$$N := \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & \ddots & & \ddots & \ddots & & \vdots \\ & & \ddots & & \ddots & \ddots & 0 \\ & & & 0 & \cdots & 0 & 1 \\ & & & & 0 & \cdots & 0 \\ & & & & & \ddots & \vdots \\ & & & & & & 0 \end{pmatrix}, \quad E := \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -c_{r-1} & c_r & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & & & \vdots \\ -c_1 & 0 & \cdots & c_r & 0 & \cdots & 0 \end{pmatrix}.$$

Here, the last r-rows of N contain only zeros.

There exist vectors $b_1, \ldots, b_r \in \mathrm{Lie}(G_n)(K_\infty)$ so that $\mathrm{Exp}_{G_n}(b_i) \in G_n(A)$ for each $1 \leq i \leq r$ such that if we set

$$b_i := [b_{i,1}, \ldots, b_{i,rn+r-1}]^{\mathsf{T}}$$

There exist vectors $b_1, \ldots, b_r \in \mathrm{Lie}(G_n)(K_\infty)$ so that $\mathrm{Exp}_{G_n}(b_i) \in G_n(A)$ for each $1 \leq i \leq r$ such that if we set

$$b_i := [b_{i,1}, \ldots, b_{i,rn+r-1}]^{\mathsf{T}}$$

and consider

$$R := \begin{pmatrix} b_{1,rn} & \ldots & b_{r,rn} \\ \vdots & & \vdots \\ b_{1,rn+r-1} & \ldots & b_{r,rn+r-1} \end{pmatrix} \in \mathrm{GL}_r(K_\infty),$$

There exist vectors $b_1, \ldots, b_r \in \mathrm{Lie}(G_n)(K_\infty)$ so that $\mathrm{Exp}_{G_n}(b_i) \in G_n(A)$ for each $1 \leq i \leq r$ such that if we set

$$b_i := [b_{i,1}, \ldots, b_{i,rn+r-1}]^{\mathsf{T}}$$

and consider

$$R := \begin{pmatrix} b_{1,rn} & \ldots & b_{r,rn} \\ \vdots & & \vdots \\ b_{1,rn+r-1} & \ldots & b_{r,rn+r-1} \end{pmatrix} \in \mathrm{GL}_r(K_\infty),$$

then $L(\phi, n+1) = c\det(R)$ for some $c \in K \setminus \{0\}$

Proving our result is now equivalent to proving the following theorem.

Theorem (Gezmiș-N., 2021)

Let $m \geq 1$. For $1 \leq \ell \leq m$, let $\boldsymbol{y}_\ell = [y_{\ell,1}, \ldots, y_{\ell,rn+r-1}]^\mathsf{T} \in \mathrm{Lie}(\mathfrak{G}_n)(\mathbb{C}_\infty)$ be such that $\mathrm{Exp}_{\mathfrak{G}_n}(\boldsymbol{y}_\ell) \in \mathfrak{G}_n(\overline{K})$. If

$$\mathcal{Y}_n := \mathbf{f}(y_{1,rn}, \ldots, y_{1,rn+(r-1)}, \ldots, \ldots, y_{m,rn}, \ldots, y_{m,rn+(r-1)}) \neq 0$$

for some non-constant polynomial $\mathbf{f} \in \overline{K}[X_1, \ldots, X_{rm}]$, then it is transcendental over $\overline{K}$.

Proving our result is now equivalent to proving the following theorem.

Theorem (Gezmiș-N., 2021)

Let $m \geq 1$. *For* $1 \leq \ell \leq m$, *let* $\boldsymbol{y}_\ell = [y_{\ell,1}, \ldots, y_{\ell,rn+r-1}]^{\mathsf{T}} \in \mathrm{Lie}(\mathfrak{G}_n)(\mathbb{C}_\infty)$ *be such that* $\mathrm{Exp}_{\mathfrak{G}_n}(\boldsymbol{y}_\ell) \in \mathfrak{G}_n(\overline{K})$. *If*

$$\mathcal{Y}_n := \mathbf{f}(y_{1,rn}, \ldots, y_{1,rn+(r-1)}, \ldots, \ldots, y_{m,rn}, \ldots, y_{m,rn+(r-1)}) \neq 0$$

for some non-constant polynomial $\mathbf{f} \in \overline{K}[X_1, \ldots, X_{rm}]$, *then it is transcendental over* $\overline{K}$.

We prove this using Chang's and Papanikolas's method, and Papanikolas's transcendence theory.

THANK YOU!